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Topological Characterization of Families of Graphs Generated by Certain Types of Graph Grammars*

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A precise topological characterization of classes of graphs generated by certain context free graph grammars (CFGG) is given. An informal interpretation of the results is that CFGG cannot generate any interesting classes of graphs besides trees and series-parallel networks, except in trivial ways (e.g. by including subgraphs of higher connectivity explicitly in the production rules).

I. INTRODUCTION

Graph grammars have been the subject of a number of papers, especially in the context of pattern recognition and picture description (Rosenfeld and Pfaltz, 1969; Shaw, 1970; Montanari, 1970; Pavlidis, 1972; 1977; Mylopoulos, 1972; Abe et. al., 1973; Cook, 1974; Fu, 1974; Della Vigna and Chezzi, 1978). A profound difference between graph and string grammars is that the languages generated by the former must be characterized, not only by "counting" criteria, but also by topological constrains on the graphs. In particular, graphs which are generated by linear or context free graph grammars must be weakly connected in the sense that not only the graphs be of low connectivity (Harary, 1969) but also have subgraphs of low connectivity (Pavlidis, 1972; 1977; Abe et. al., 1973; Della Vigna and Chezzi, 1978).

Shaw (1970) proposed the use of blanking operators and labels in order to circumvent this problem. For example, the complete graph of four nodes (K_4) can be generated by a context free grammar, by first generating the graph shown in Fig. 1a, and then matching nodes A to A' and D to D' and blanking the branches BA' and $D'C$. Unfortunately such rules make parsing difficult. In the absence of such operations, if H is a subgraph whose production has been found, it need not be revisited during the rest of the parsing procedure. This will not be the case during parsing under grammar using blanking operators. The following is a rather general definition of context-free graph grammars (Pavlidis, 1972).

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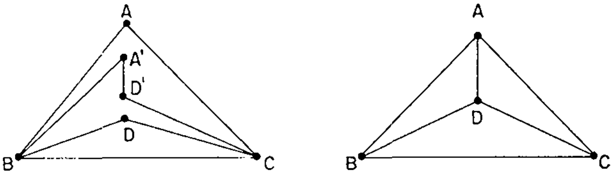

(a) (b)

FIG. 1. Generation of K_4 by Shaw's context free picture description grammar.

DEFINITION. A 2^d *order context-free graph grammar* (CFGG) is a quadruple (N, T, P, I) where

- N is a set of nonterminal elements: node structures and branch structures,
- T is a set of terminal elements: nodes and edges,
- P is a (finite) set of productions (rewriting rules) of the form $G \to H$, where $G \in N$ and H is a graph containing possibly both terminals and nonterminals,
- H is connected to the rest of the graph through exactly the same nodes as G,
- I is the set of initial graphs.

The terms node and branch structures refer to structures connected to the rest of the graph by a single node or a pair of nodes. One could also think of them as nodes and branches labeled as nonterminal.

In this paper we present characterizations for graphs generated by such grammars and also by the grammar proposed by Shaw without the blanking operation and labeling. His notation makes convenient the use of string symbols, which simplifies the presentation of the proofs. The grammar is described formally as following:

$$
\begin{aligned}
&PDG/1 = (V_N, V_T, P, I),\\
&V_N = \{B, Q\},\ V_T = \{+, \times, -, *, b\}, I = \{B\},\\
&B \to b/{\sim}B/BQB,\\
&Q \to +/\times/-/*.
\end{aligned}
$$

The strings produced by this grammar are mapped into (directed) graphs by using the following convention. b is a branch with a *head* and a *tail*. Fig. 2 shows how such branches are linked with the elements of Q and the assignment of head and tail to the resulting subgraph. The operator $\sim$ reverses the role of head and tail.

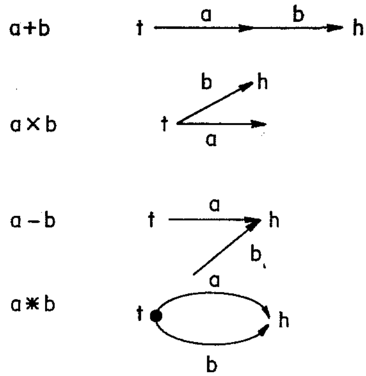

FIG. 2. Interpretation of the operations of PDL/1.

If we think of B also as a branch, then we can construct an equivalent graph grammar with the following rules:

Insert a node in a branch.
Replace a branch by a parallel pair of branches.
Attach a branch to a node.

We have to ignore the assignment of head and tail, but this does not increase the power of the grammar.

In the sequel we shall limit the term "graph" to mean an undirected graph with, possibly, multiple edges connecting any two nodes. We shall show that $PGD/1$, in effect, can generate only series-parallel networks (SPN), or trees or trees whose brances are SPN. We will express the conditions in terms of undirected graphs. Thus, if we say that a graph G is generated by $PDG/1$, it will mean that there exists an assignment of directions on edges, as well as head and tail, to make G a directed graph generated by $PDG/1$. We denote by $PDL/1$ the corresponding language, i.e., the set of (undirected) graphs generated by $PDG/1$.

In Section 3 we consider general 2^d order context-free graph grammars and present certain characterizations for the families of graphs produced by them.

II. Characterization of Graphs in PDL/1

For two nodes u and v of a graph $G = (N, E)$, let $\mu_G(u, v)$ denote the maximum number of node-disjoint paths connecting u and v, with multiple edges counted as a single path.

Define a *property* P on graphs as follows: A graph G has the property P if for every two nodes u and v, $\mu_G(u, v) \geqslant 3$ implies that u and v separate nodes lying on disjoint $u - v$ paths. That is, if x and y are two nodes for which there exist disjoint paths $u - x - v$ and $u - y - v$, then x and y belong to different connected components of $G - \{u, v\}$.

We will show that property P is a necessary and sufficient condition for a connected graph to be in $PDL/1$. At first we will prove the sufficiency part in two lemmas. In Lemma 1 we show how to construct a derivation in $PDG/1$ of a biconnected graph with property P. In Lemma 2 we show how to combine the derivations of the blocks (biconnected components) of a graph G with property P, in order to get a derivation in $PDG/1$ of the whole graph G.

Note here that P is a hereditary property, i.e., if G is a grapgh with property P and H any subgraph of it, then H has also property P. The reason is that every path in H is also a path in G.

LEMMA 1. *Let G be a biconnected graph with property P and t, h two nodes of it. If t and h separate their node-disjoint paths, then there is a derivation of G in PDG/*1 *with t as the tail and h as the head.*

Proof. By induction on the number of edges of G. The basis is trivial.

Let $H_i' = (N_i', E_i')$ $(i = 1,..., r)$ be the connected components of $G - \{t, h\}$ and $H_i = (N_i, E_i)$ $(i = 1,..., r)$ the subgraphs of G, induced by $N_i' \cup \{t, h\}$ without the edges $\{t, h\}$ if there are any in G. In the last case we have s more components $H_{r+1},..., H_{r+s}$, if there are s edges joining t and h, each additional component consisting of a single edge.

Clearly G is the edge-disjoint union of the H_i's. The H_i's are singly connected $(k(H_i) = 1)$, $i = 1,..., r$, since otherwise there must be two node-disjoint $t - h$ paths p_1, p_2 in H_i. But then $p_1' = p_1 - \{t, h\}$ and $p_2' = p_2 - \{t, h\}$ belong to H_i' which is connected and therefore $\{t, h\}$ do not separate p_1 and p_2, contradicting our assumption about the selection of t and h. Consequently each H_i must have at least one edge less than G. It can be shown that the block-graph $N(H_i)$ of H_i $(i = 1,..., r)$ is a path (see *Claim* 1 in the Appendix). Let B_j be the jth block in $B(H_i)$, t_j its common node with B_{j-1}, h_j with B_{j+1}. Then $t_j \neq h_j$ because otherwise B_{j-1} would be adjacent to B_{j+1} in the block graph.

Let p_1, p_2 be two node-disjoint paths joining t_j, h_j in B_j. If there is a path connecting a node u of p_1 to a node v of p_2 not passing through t_j, h_j, then $\mu_G(u, v) \geqslant 3$ and the paths $u -^{p_1} h_j -^{p_2} v$, $u -^{p_1} t_j -^{p_2} v$ (see Fig. 3) are joined through t, h and the rest of the graph G, contradicting our assumption, that G has property P. Therefore t_j and h_j separate their disjoint paths and by the

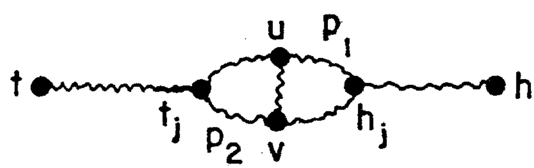

FIG. 3. Illustration used in the proof of lemma 1.

inductive hypothesis B_j can be derived with t_j as the tail and h_j as the head. Then using the $+$ operation we can connect the blocks of H_i and derive it with (t, H) as (tail, head). G can be derived from the H_i's using the $*$ operation. Q.E.D.

(Note that in the above derivation only the operations $+$ and $*$ were used).

Remark. Lemma 1 implies in particular that if G is a biconnected graph with property P and t any node of it, then there is a derivation of G in the grammar with t as the tail, because we can take as the head h any node adjacent to t. Then if there are two non-trivial paths connecting t and h, $\mu_G(t, h) \geqslant 3$, and $\{t, h\}$ must separate them, since G has property P.

LEMMA 2. *If G is a connected graph with property P and t any node of it, then there is a derivation of G in $PDG/1$ with t as the tail.*

Proof. We will use induction on the number of biconnected components of G. The basis follows from the previous remark.

For the inductive step, let B be the biconnected component that contains t, $B_1, \ldots, B_n$ the blocks having common nodes $t_1, \ldots, t_n$ respectively with B (not necessarily distinct), and $G_1, \ldots, G_n$ the subgraphs of G, where G_i contains B_i and the blocks connected with it in $G - B$. Clearly G is the edge-disjoint sum of B and the G_i's. For $i \neq j$, G_i and G_j cannot have a node v in common that does not belong to B (because then u_i and v would lie on a common cycle).

By inductive hypothesis there is a derivation of G_i in the grammar with t_i as the tail, since P is a hereditary property. For each t_i let $(t_i, h_i) = b_i$ be an edge outgoing from t_i in B with the orientation defined on the edges of B as in the proof of Lemma 1 (i.e. disregarding the other biconnected components). Then $b_i \times G_i$ adds correctly the subgraph G_i to B; i.e. if $S \overset{*}{\to} \alpha S \beta \to \alpha b_i \beta \overset{*}{\to} B$ is a derivation of B, then $S \overset{*}{\to} \alpha S \beta \to \alpha(S \varnothing S)\beta \to \alpha(S \times S)\beta \to \alpha(b_i \times S)\,\beta \overset{*}{\to} \alpha(b_i \times G_i)\,\beta \overset{*}{\to} B'$ is a derivation of $B' = B \cup G_i$, and in a similar way we can add also the other G_i's. Q.E.D.

THEOREM 1. *A graph G belongs to $PDL/1$ if and only if it is connected and has property P.*

Proof. (a) *Necessity*: It is obvious that $PDG/1$ produces only connected graphs. Let G be a graph in $PDL/1$. We will prove by induction on the length of a derivation, that the graph G' obtained from G by adding the edge $\{t, h\}$ has property P. From this it will follow the that G must also have property P, since P is a hereditary property.

The basis (G is a primitive) is trivial.

For the inductive step we consider 3 cases according to the operation last applied:

(1) The last operation is $\sim$, x or $-$. The result follows trivially from the inductive hypothesis.

(2) The last operation is $+$. Let $G = G_1 + G_2$ with t_1, h_1 and t_2, h_2 the tail and the head of G_1 and G_2 respectively. We have: $t = t_1$, $h = h_2$ and $t_2 = h_1$. If ($u \in G_1$ and $v \in G_2$) or ($u \in G_2$ and $v \in G_1$) then $\mu_G(u, v) = 1$ (since every $u - v$ path has to pass through $t_2 = h_1$). Therefore $\mu_{G'}(u, v) \leqslant 2$.

If u and v belong to the same component, say G_2, then every $u - v$ path in G, is either contained in G_2, or uses the edge $\{t, h\}$ (see Fig. 4a). For a path p of the latter kind, there exists another path p' in G_2', which uses the edge $\{h_2, t_2\}$ and exactly the same nodes of G_2 as p. Therefore, if $\mu_{G'}(u, v) \geqslant 3$, then also $\mu_{G_2'}(u, v) \geqslant 3$, and if u and v do not separate two disjoint $u - v$ paths in G', the same holds also for G_2'. Thus the conclusion follows from the inductive hypothesis for G_2'.

(3) The last operation is $*$. Let $G = G_1 * G_2$ (see Fig. 4b). If ($u \in G_1$ and $v \in G_2$) or ($u \in G_2$ and $v \in G_1$), then $\mu_{G'}(u, v) \leqslant 2$, since a $u - v$ path must pass either through t or through h. If u and v belong to the same component, say G_1, there can be only one new path in G (and G') joining them: passing through h, G_2 and t. (see Fig. 4b). But then there is a corresponding path in G_1 using the edge $\{t, h\}$. The conclusion follows from the inductive hypothesis for G_1'.

(b) *Sufficiency*: Follows from Lemma 2. Q.E.D.

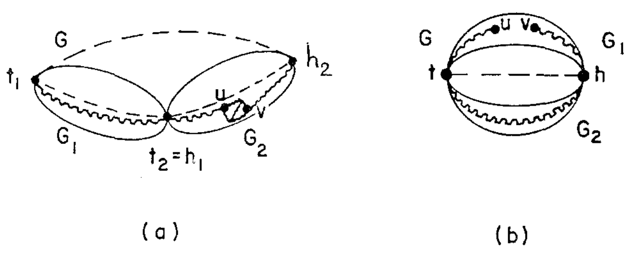

FIG. 4. Illustrations for the proof of necessity in theorem 1.

Remarks. (1) In the above derivation we used only the three operations $+$, $\times$, $*$. We implicitly assumed however, that if e is an edge then we can assign arbitrarily an orientation to it; i.e., we can restrict the application of $\sim$ only to primitives. The fact that $-$ is redundant can be seen directly by observing that $a - b = \sim((\sim b) \times (\sim a))$ disregarding orientations.

(2) Lemmas 1 and 2 give also an $O(\| E \|)$ algorithm for the construction of a derivation of a graph $G = (N, E)$ in $PDG/1$, if there exists one.

COROLLARY. *If a graph G belongs to the $PDL/1$, then $1 \leqslant k(G) \leqslant 2$, where $k(G)$ is the connectivity of G.*

Proof. (1) Suppose G is complete: $G = K_p$. If $p \leqslant 3$, then $k(G) \leqslant 2$. If $p \geqslant 4$, let v_1, v_2, v_3, v_4 be four nodes of G. Then $\mu(v_1, v_4) \geqslant 3$ ($v_1 - v_2 - v_4$, $v_1 - v_3 - v_4$, $v_1 - v_4$ are 3 disjoint $v_1 - v_4$ paths) and there is an edge $\{v_2, v_3\}$. Therefore G doesn't have property P.

(2) Suppose that G is not complete. Then by Menger's theorem $k(G) = \min \mu_G(u, v)$, the minimum taken over all pairs of nonadjacent nodes. Let u, v be two nonadjacent nodes. If $\mu_G(u, v) \leqslant 2$, then $k(G) \leqslant 2$. If $\mu_G(u, v) \geqslant 3$, then u and v separate their disjoint paths (by property P) and hence form a cutset of G. Therefore $k(G) \leqslant 2$. Q.E.D.

DEFINITIONS (Harary, 1969). (1) Two graphs G_1 and G_2 are *homeomorphic* if both can be obtained from the same graph H by a sequence of subdivisions of it edges. (An edge $e = \{u, v\}$ of H is subdivided when it is replaced by two edges $\{u, w\}$ and $\{w, v\}$, with w a new node, not in H.)

(2) An *elementary contraction* of a graph G is obtained by identifying two adjacent nodes u and v. That is, we replace u and v by a new node w adjacent to those nodes to which u or v was adjacent.

A graph G is *contractible* to a graph H if H can be obtained from G by a sequence of elementary contractions. Equivalent necessary and sufficient conditions for a graph G to belong to the $PDL/1$ in terms of forbidden homeomorphisms and contractions will be given using the next lemma.

LEMMA 3. *For a connected graph G the following are equivalent:*

(1) *G does not possess property P.*

(2) *G has a subgraph homeomorphic to K_4.*

(3) *G is contractible to K_4.*

Proof. (1) $\to$ (2): There are two nodes u, v with $\mu_G(u, v) \geqslant 3$ and a path p connecting two nodes x, y lying on two different paths, say p_1, p_2 (see Fig. 5). Let H be the subgraph of G formed by the paths p_1, p_2, p_3, p. Clearly H is homeomorphic to K_4.

(2) $\to$ (3): Obvious

(3) $\to$ (1): We showed in the proof of the corollary that K_4 does not have property P. Let H_1 be a graph transformed through an elementary contraction to H_2. We will show that if H_2 does not have property P, then the same holds also for H_1. From this it follows that G does not have property P.

Suppose that u, v are contracted to w, and that there exist $x, y \in H_2$ such that $\mu_{H_2}(x, y) \geqslant 3$ and x, y do not separate their disjoint paths. We note that for each

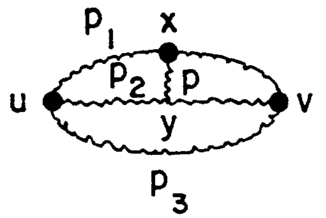

FIG. 5. Illustration used in the proof of lemma 3.

path p in H_2, there is a path p' in H_1 having exactly the same nodes with p, except if $w \in p$, in which case either u or v, or both to p'.

Case 1. $w \neq x, y$. Let p_1, p_2, p_3 be three disjoint $x - y$ paths in H_2 with p a crosspath joining, say, p_1 to p_2. Then p_1', p_2', p_3' are also disjoint in H_1 and p' joins p_1' to p_2'. (Note that at most one of the paths p_1, p_2, p_3 passes through w).

Case 2. $w = x$. Let s, t be the endpoints of a crosspath p having no other points in dommon with p_1, p_2, p_3. (There exist always such a crosspath). Then $\mu_{H_2}(s, t) \geqslant 3$ (p, $s -^{p_1} w -^{p_2} t$, $s -^{p_1} y -^{p_2} t$) are three disjoint $s - t$ paths—see Fig. 6) and s, t don't separate the two last paths. Therefore by case 1, H_1 does not have also property P. Q.E.D.

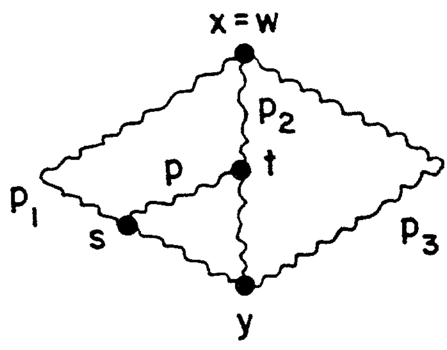

FIG. 6. Illustration used in the proof of lemma 3, case 2.

THEOREM 2. *For a graph G the following are equivalent*:

(1) *G belongs to PDL/*1.

(2) *G is connected and has no subgraph homeomorphic to* K_4.

(3) *G is connected and is not contractible to* K_4.

Proof. By Lemma 3 and Theorem 1.

III. Characterization of Graphs Generated by 2^d Order CFGG

We proceed now with the more general case where the rewriting rules allow the replacement and/or attachment of arbitrary subgraphs rather than just branches. First we should point out that when we say that a set of graphs S can be generated by a *CFGG* we do not mean that there is a grammar G generating exactly S but that S is a subset (possibly proper) of the set of graphs generated by G. This distinction is necessary in order to emphasize the topological characterizations. Otherwise we may have a situation where a family of weakly connected graphs could not be generated by a *CFGG* because of counting arguments. For example this is the case with the family of graphs described in Fig. 7. The

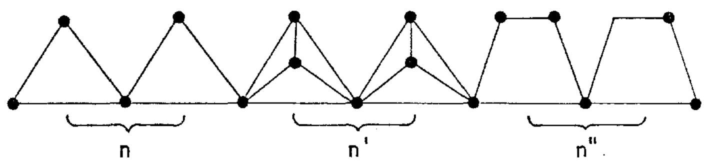

FIG. 7. A member of a class of graphs characterized by the property $n = n' = n''$.

restriction $n = n' = n''$ suggests that this family cannot be exactly generated by a *CFGG* (Hopcroft and Ullman, 1969), but it can be produced as a subset of a family S' by another grammar where the above constraint is not imposed (see also Cook, 1973).

Note also that under the formalism of *CFGG*, it is possible to construct graphs with arbitrarily high (but finite) connectivity in a trivial manner, by having them as the result of a single production rule. (This was not possible under the formalism of *PDL*/1).

Let R_k be the 2^d order *CFGG* which contains all possible productions whose R.H. side has not more than k nodes or node structures, and L_k the corresponding language (set of graphs). Clearly $L_k \leqslant L_l$ for $k \leqslant l$, and $PDL/1 \subseteq L_3$.

We will start showing our characterization by proving first the sufficiency part for biconnected graphs.

LEMMA 4. *Let G be a biconnected graph and x, y two nodes of it. If G', the graph obtained from G by adding the edge $\{x, y\}$ has no subgraph homeomorphic to a 3-connected graph with more than k nodes, then G is derivable in R_k from a branch structure between x and y.*

Proof. We use double induction: on k and n, the number of nodes of G. The basis for k ($k = 3$) follows from Theorem 2. (The smallest 3-connected graph has 4 nodes and is K_4. Also $PDL/1 \subseteq L_3$.) For any k, the basis for n is trivial.

Now let G be a biconnected graph with n nodes satisfying the hypotheses of the lemma with $k \geqslant 4$. If G' has no subgraph homeomorphic to a 3-connected graph with k nodes, then the conclusion follows from the inductive hypothesis on k, since $L_{k-1} \subseteq L_k$. So assume that there exists such a subgraph H with nodes $u_1, \ldots, u_k$ of degree $\geqslant 3$, and interconnecting paths $p_1, \ldots, p_r$. It can be shown (see *Claim* 2 in the Appendix) that each path p_i is separated from the rest of H by its endpoints.

Let u_{i_1} and u_{i_2} be the endpoints of a path p_i. Net N_i be the set of nodes of the connected components of $G' - \{u_{i_1}, u_{i_2}\}$ that do not contain any node u_j, and let G_i be the subgraph of G induced by $N_i \cup \{u_{i_1}, u_{i_2}\}$. By Claim 2 the subgraph G_i are disjoint aside from nodes $u_1, \ldots, u_k$, and cover the whole graph G'. Because of the edge $\{x, y\}$, the nodes x and y belong to the same G_i, say G_1, with endpoints u_1, u_2 in H (x, y need not be distinct from them) (Fig. 8). It is clear that every G_i satisfies the assumptions of the lemma with respect to its endpoints[1] and has fewer nodes that G. Thus, by the inductive hypothesis on n, it can be derived in R_k from a branch structure between its endpoints.

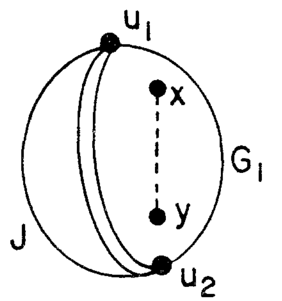

FIG. 8. Illustration used in the proof of lemma 4.

Consequently, if $J = G' - G_1$, then J can be derived from a graph $\hat{H}$ with k nodes $u_1, \ldots, u_k$, with branch structures in place of its edges. ($\hat{H}$ is the 3-connected graph homeomorphic to H). Since R_k contains all productions whose R.H. sides have at most k nodes or node structures, H can be in turn derived from a branch structure between u_1 and u_2. If G_1 is G_1 plus the edge $\{u_1, u_2\}$ then obviously G_1 is bicoonected and satisfies the assumptions of the lemma with respect to x and y. Thus the conclusion follows from the inductive hypothesis on n. Q.E.D.

We are ready now to prove our characterization.

THEOREM 3. *A family of graphs S is generated by a 2^d order CFGG iff there exists a number k depending only on S such that every member of S has no subgraph homeomorphic to a 3-connected graph with more than k nodes.*

[1] If a component G_i is not biconnected then its block graph is a path (see Claim 1 in the Appendix), and the derivation in R_k of G_i proceeds through its blocks as in lemma 1.

Proof. (a) *Necessity*: The proof is analogous to the proof of the necessity part in Theorem 1.

Let k be the maximum number of nodes and node structures in the R.H. sides of the productions of a 2^d order *CFGG* generating S. We use induction on the length of a derivation with branch structures replaced by edges and node structures by nodes. The basis is trivial.

(i) Production of the form $N \rightarrow G_1$; N a node structure (Fig. 9a). If there were such a subgraph H, it should be totally contained in G' or in G_1, since N is a cutnode. The first case cannot happen by the inductive hypothesis and the second because of our choice of k.

(ii) Production of the form $B \rightarrow G_1$; B a branch structure, (Fig. 9b). If V is the set of nodes of degree $\geqslant 3$ of such a forbidden subgraph H, then we must have again either $V \subseteq G'$, or $V \subseteq G_1$, since $\{u, v\}$ constitute a cutset of the graph. The first case cannot happen by the inductive hypothesis (note that a path in G can be simulated in $G' \cup B$ using the edge B) and the second by the choice of k.

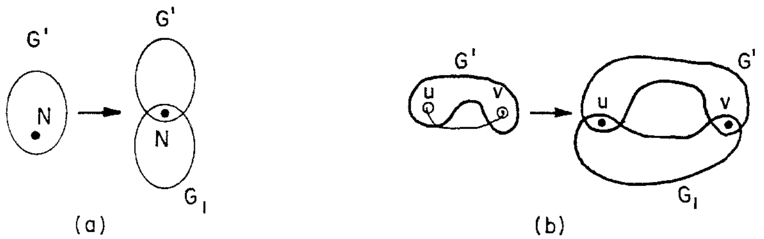

FIG. 9. Illustration used in the proof of theorem 3.

(b) *Sufficiency.* Let G be a graph in the family S. Suppose first that G is biconnected, x is any node of it and y any node adjacent to x. It follows from Lemma 4 that G can be derived in R_k from a branch structure between x and y. For G a singly connected graph, the proof of the theorem is similar to the proof of Lemma 2, i.e., proceeds inductively on the number of biconnected components of G, where now the productions of R_k with a node structure in the L.H. side replace the x operation of the *PDG*/1. Q.E.D.

As in the case of the *PDL*/1, an equivalent characterization in terms of contractibility will be given using the next lemma:

LEMMA 5. *If G is a connected graph, then G has a subgraph homeomorphic to a 3-connected graph with at least k nodes iff G is contractible to a 3-connected graph with k nodes.*

Proof. ($\Rightarrow$) Obvious, considering that G is connected and the homeomorphic graph will have no nodes of degree $\leqslant 2$ since it is 3-connected.

($\Leftarrow$) Working as in the proof of Lemma 3, let H_1 be a graph transformed through an elementary contraction to H_2, and suppose that H_2 has a subgraph J_2 homeomorphic to a 3-connected graph with k nodes. We will show that H_1 has a corresponding subgraph J_1 homeomorphic to a 3-connected graph with at least k nodes. Let u, v be the adjacent nodes of H_1 contracted to w in H_2, and N the set of nodes of degree $\geqslant 3$ of J_2.

Case 1. $w \notin N$. Replacing a path p in H_2 by a path p' in H_1, as described in the proof of Lemma 3, the subgraph $J_2 = J_1'$ of H_1 is homeomorphic to a 3-connected graph with k nodes (the elements of N_1).

Case 2. $w \in N$. Then $\deg(w) \geqslant 3 \Rightarrow \deg(u) + \deg(v) \geqslant 5$. (here the degree is w.r. to J_1, J_2.) If $\deg(u) = 2$, i.e. u is connected to $H_1 - \{u, v\}$ only through one edge $\{u, x\}$, then the subgraph J_1 obtained from J_2 by replacing w by v and the edge (w, x) by the path $v - u - x$ in H_1, is homeomorphic to the same graph as J_2. Similarly if $\deg(u) = 2$.

If $\deg(u) \geqslant 3$ and $\deg(v) \geqslant 3$, then $\deg(w) \geqslant 4$. Let J_1 be obtained from J_2 by replacing w by both nodes u, v and the corresponding edges. Then J_1 is homeomorphic to a 3-connected graph with $k + 1$ nodes (Tutte, 1966; Harary, 1969).

Remark. The lemma is in general false if we omit the "at least". A simple counterexample is shown in Fig. 10 with $k = 5$. G has no subgraph homeomorphic to a 3-connected graph with 5 nodes (though G itself is 3-connected and has 6 nodes), whereas it is contractible to the wheel W_4.

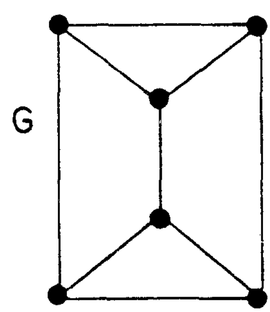

FIG. 10. Illustration used in the proof of lemma 5.

THEOREM 4. *A family of graphs S is generated by a 2^d order CFGG iff there exists a number k depending only on S such that no member of S is contractible to a 3-connected graph with k nodes.*

Proof. By Lemma 5 and Theorem 3.

IV. Conclusions

We have presented some concise topological characterizations for families of graphs generated by Context Free Graph Grammars. Theorem 2 expresses these conditions in terms of homeomorphism or contractibility to K_4, the complete graph of four nodes. Theorems 3 and 4 express them in terms of homeomorphism or contractibility to a 3-connected graph with no more than a fixed number of nodes. Earlier characterizations either dealt with specific families which could or could not be generated by *CFGG* (Abe et. al., 1973) or proved the existence of procedures for verifying certain properties of the graphs (Pavlidis, 1972; Della Vigna and Chezzi, 1978). Since homeomorphism and contractibility are well defined graph properties, there is some advantage presenting characterizations through them.

Appendix

Claim 1. *Let G be a biconnected graph with $\{t, h\}$ a cutset. Let $H' = (N', E')$ be a connected component of $G - \{t, h\}$ and $H = (N, E)$ the subgraph of G induced by $N = N' \cup \{t, h\}$. The block graph $B(H)$ of H is a path.*

Proof. (1) At first we note that t belongs only to one block, because otherwise the removal of t would disconnect H contradicting the fact that H' is connected.

Suppose that the block B_t containing t is adjacent to two blocks B_1, B_2, with common nodes u_1, u_2 (not necessarily distinct) and let v_1, v_2 be two other nodes of B_1, B_2 respectively (Fig. 11a). Since G is biconnected, there is a path in G from v_1 to t not passing through u_1 and similarly for v_2. Since v_1 and t, v_2 and t belong to different blocks of H, these two paths must pass through h, which means that there is a cycle in H containing v_1 and v_2 and therefore that v_1 and v_2 should belong to the same block. The same can be similarly proved for h.

(2) Let B_j be a block different from B_t, B_x. We will prove that $\deg(B_j) = 2$. Clearly, since H is connected, $\deg(B_j) \geqslant 1$. Suppose that $\deg(B_j) = 1$ and let u_j be its common node with another block. Then clearly u_j is a cutnode in G, contrary to our assumption that G is biconnected.

Suppose that $\deg(B_j) \geqslant 3$, and let B_1, B_2, B_3 be three blocks adjacent to B_j with common nodes u_1, u_2, u_3 (not necessarily distinct) and v_1, v_2, v_3 three other nodes of them (Fig. 11b). Then as in (1) there is a path in H from v_k either to t or to h, not passing through u_k ($k = 1, 2, 3$).

But then at least two of $\{v_1, v_2, v_3\}$ will lie on a common cycle in H and therefore should belong to the same block. Q.E.D.

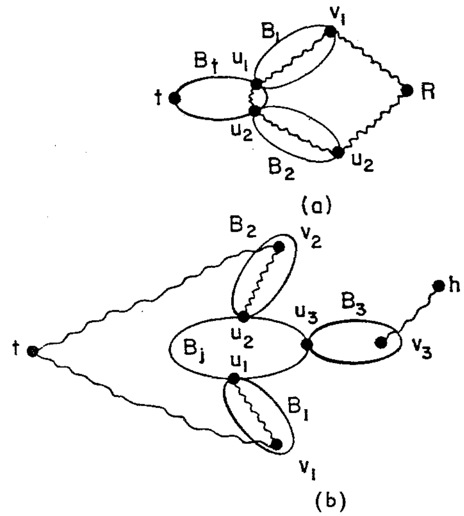

FIG. 11. Illustration used in the proof of the claim 1.

CLAIM 2. *Let G be a graph with no subgraph homeomorphic to a* 3-*connected graph with more than k nodes. Suppose that H is a subtraph of G homeomorphic to a* 3-*connected graph H with k nodes, $u_1, \ldots, u_k$ the nodes of H of degree $\geqslant 3$ and $p_1, \ldots, p_r$ the interconnecting paths. Then each path p_i is separated (in G) from the rest of H by its endpoints.*

Proof. Suppose to the contrary that the path p_1 connecting nodes u_1 and u_2 is not separated from the rest of H by them. Then there must be a crosspath p connecting p_1 to another path p_j, $1 < j \leqslant r$, and having no nodes in common with H. Let u be the common node of p and p_1, and v of p and p_j (v may coincide with an endpoint of p_j, provided it is not u_1 or u_2). Then $H \cup p$ is homeomorphic to a graph H_1 having at least $k + 1$ nodes, and H_1 is 3-connected by a theorem of Tutte (1966). Q.E.D.

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